

# Profit max problems

## Solutions

$$1) F(L, K) = L^{\frac{1}{2}} K^{\frac{1}{4}} \quad \frac{1}{2} + \frac{1}{4} L \text{ drts}$$

- For  $L^{SR}$

$$MP_L = w \quad \checkmark$$

$$\cdot \max \Pi: P \cdot L^{\frac{1}{2}} K^{\frac{1}{4}} - wL - rK \quad MP_K = r \quad \checkmark$$

- FOC (derivative in terms of  $L$ , since we want to find the amount of  $L$  that maximizes profit):

$$\frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} \cdot K^{\frac{1}{4}} - w = 0 \rightarrow \frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} \cdot K^{\frac{1}{4}} = w \rightarrow$$

(This is  $P \cdot MP_L = w$ )

$$L^{-\frac{1}{2}} = \frac{w}{\frac{1}{2} \cdot P \cdot K^{\frac{1}{4}}} \rightarrow \boxed{L^{SR} = \frac{P^2 \cdot K^{\frac{1}{2}}}{4 \cdot w^2}}$$

- For  $L^{LR}$

$$\cdot \max \Pi: P \cdot L^{\frac{1}{2}} \cdot K^{\frac{1}{4}} - wL - rK$$

Since  $K$  is no longer fixed, we need to get rid of it to solve for  $L$ . We use tangency. We did not use tangency before because the trs is null

$$\cdot FOC: \frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} \cdot K^{\frac{1}{4}} = w \leftarrow$$

(there is no "rate of substitution" when one variable is fixed)

$$\cdot \text{trs} = \frac{MP_L}{MP_K} = \frac{2K}{L}$$

$$\cdot \frac{2K}{L} = \frac{w}{r} \rightarrow K = \frac{wL}{2r}$$

$$\cdot \frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} \cdot \left(\frac{wL}{2r}\right)^{\frac{1}{4}} = w \rightarrow$$

to find  $K^{LR}$ , plug  $L^{LR}$  into the tangency condition

$$\frac{\frac{1}{2} \cdot P \cdot L^{-\frac{1}{4}} \cdot w^{\frac{1}{4}}}{2^{\frac{1}{4}} \cdot r^{\frac{1}{4}}} = w \rightarrow \frac{1}{2} \cdot P \cdot L^{-\frac{1}{4}} \cdot w^{\frac{1}{4}} = w \cdot 2^{\frac{1}{4}} \cdot r^{\frac{1}{4}} \rightarrow$$

$$L^{-\frac{1}{4}} = \frac{w \cdot 2^{\frac{1}{4}} \cdot r^{\frac{1}{4}}}{\frac{1}{2} \cdot P \cdot w^{\frac{1}{4}}} \rightarrow$$

$$\boxed{L^{LR} = \frac{P^4}{32 \cdot w^3 \cdot r}}$$

$$\boxed{K^{LR} = \frac{P^4}{64w^2r^2}}$$

$$2) F(L, K) = L^{\frac{1}{2}} + K \quad P=10 \quad w=5 \quad r=12$$

- For  $L^{SR}$   $L$  has drts,  $K$  has crts, the function has drts

- Max  $\pi$ :  $P(L^{\frac{1}{2}} + K) - wL - rK$

- FOC:  $\frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} = w \rightarrow L^{\frac{1}{2}} = \frac{w}{\frac{1}{2} \cdot P} \rightarrow$

$$\begin{aligned} L^{SR} &= \frac{P^2}{4w^2} \\ &= \frac{100}{4(25)} = 1 \end{aligned}$$

- For  $K^{LR}$

Crts,  $K=0$  or  $K=\infty$ . The Profit/Loss we make on the first unit will be the profit/loss we make on all units.

- $K=1 \quad P \cdot Q = wL = rK \quad \text{lose \$2 for every additional unit of capital.}$   
 $10 \cdot 1 = 0 - 12(1) = -2$
- $K=2 \quad 10 \cdot 2 = 0 - 12(2) = -4$

- $K^{LR} = 0$

- For  $L^{LR}$

- Max  $\pi$ :  $P \cdot L^{\frac{1}{2}} - wL - rK$

- FOC:  $\frac{1}{2} \cdot P \cdot L^{-\frac{1}{2}} - w = 0 \rightarrow L^{-\frac{1}{2}} = \frac{w}{\frac{1}{2} \cdot P} \rightarrow$

$$L^{LR} = \frac{P^2}{4w^2} = L^{SR}$$

- for wage elasticity of short run Labor demand

- $\epsilon_{L^{SR}, w} = \frac{dL^{SR}}{dw} \cdot \frac{w}{L^{SR}} \rightarrow -2 \cdot \frac{P^2}{4w^3} \cdot \frac{w}{\frac{P^2}{4w^2}} \rightarrow -2 \cdot \frac{P^2}{4w^3} \cdot \frac{4w^3}{P^2}$

$$\frac{P^2}{4w^2} \rightarrow P^2 \cdot 4w^{-2} \rightarrow \text{derr} \rightarrow -2 \cdot P^2 \cdot 4w^{-3}$$

$$\epsilon_{L^{SR}, w} = -2$$

$$3) F(L, K) = \min(L, \sqrt{K}) \quad P=10 \quad w=2 \quad r=1 \quad \bar{K}=16$$

$\xrightarrow{\text{drts}} \rightarrow P \cdot MP_L = w$

- For  $L^{SR}$

- $L = \sqrt{K} \rightarrow L = \sqrt{16} \rightarrow \boxed{L^{SR} = 4}$

- For  $L^{LR}$

- $\max \Pi: P \cdot \min(L, \sqrt{K}) - wL - rK$

- $\bullet L = \sqrt{K} \text{ and } K = L^2 \text{ so:}$

- $\max \Pi: P \cdot \min(L, L) - wL - r(L^2) \rightarrow$

$$\max \Pi: P \cdot L - wL - rL^2$$

- $\bullet$  why did I plug in  $L^2$  for  $K$ ? If I didn't, when we took the derivative  $r_K$  would die, and there would be no  $r$  in the answer. We know that a perfect complement needs to include the price of both goods in its demand function (ex  $X^* = \frac{m}{P_x + P_y}$ ). By substituting  $L^2$  for  $K$ , we keep  $r$  in the equation.

$$\max \Pi: P \cdot L - wL - rL^2$$

- $\text{FOC: } P - w - 2rL = 0 \rightarrow 2rL = P - w \rightarrow$

$$\boxed{L^{LR} = \frac{P - w}{2r} = \frac{8}{2} = 4}$$

- $K^{LR} = L^{LR^2} \rightarrow \boxed{K^{LR} = \left(\frac{P - w}{2r}\right)^2 = 16}$

$$4) F(L, K) = 2L + 5K \quad P=2 \quad w=5 \quad r=12 \quad \bar{K}=2$$

↑  
CRTS = ∞ or 0 unless special conditions given.

- For  $L^{SR}$

- CRTS → are we making a profit / loss on the first unit? that is the profit / loss we will make for every unit.

Rev	Cost	Profit	
P · Q	wL		lose a dollar for every additional unit of labor. so,
L=1 2 · 2	5 · 1	-1	
L=2 2 · 4	5 · 2	-2	$L^{SR} = 0$

A quicker way to do this problem would be to compare  $P \cdot MP_L$  and wage.  $P \cdot MP_L$  is the marginal revenue from a worker, and wage is the marginal cost of one more worker.

- $P \cdot MP_L = 4 \quad w = 5$  (notice they are both constant)

If I always get \$4 of revenue for one more worker, but they cost \$5, I should not hire any workers.

- For  $L^{LR}$  and  $K^{LR}$

- L makes the same loss it did in the short run, so  $L^{LR} = 0$
- we compare  $P \cdot MP_K$  vs  $r$  and get

$$P \cdot MP_K = 10 \quad r = 12$$

$$\boxed{\text{So } K^{LR} = 0}$$

$$5) F(L, K) = 4L + 2K \quad P=4 \quad w=15 \quad r=10$$

new:  $w=20$

- For  $L^R, K^R$

- Crts, perf subs. compare  $P \cdot MP_L$  vs  $w$  and  $P \cdot MP_K$  vs  $r$
- $P \cdot MP_L = 16$     $w=15$  (for 5 workers) then  $w=20$   
Hire the first 5 workers only. After the first 5, additional workers will reduce profit, so  $L^R = 5$
- $P \cdot MP_K = 8$     $r=10$     $K^R = 0$

$$6) F(L, K) = \max(2L^{\frac{1}{2}}, K) \quad P=6 \quad w=2 \quad r=8$$

- For  $K^{LR}$

- For a max function, choose all L or all K
- If all K, the function becomes  $f(L, K) = K$
- Crt's, compare  $P \cdot MP_K$  vs  $r$
- $P \cdot MP_K = 6 \quad r = 8 \quad \boxed{K^{LR} = 0}$

- For  $L^{LR}$

- if all L, function becomes  $f(L, K) = 2L^{\frac{1}{2}}$
- drts, finite answer.
- Max  $\pi$ :  $P \cdot 2L^{\frac{1}{2}} - wL - rK$
- FOC:  $P \cdot L^{-\frac{1}{2}} - w = 0 \rightarrow L^{-\frac{1}{2}} = \frac{w}{P} \rightarrow \boxed{L^{LR} = \frac{P^2}{w^2} = \frac{36}{4} = 9}$