

# Utility function handout

## Solutions

### Perf Substitutes

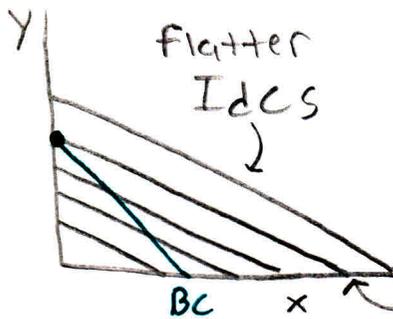
1)  $U(x, y) = 5x + 2y$      $P_x = 6$     $P_y = 2$     $M = 24$     Find  $x^*$  and  $y^*$

Subs: Consume all of the good with the better bang per buck

Method 1: Compare  $MRS$  vs  $\frac{P_x}{P_y}$

$$MRS = \frac{M_{ux}}{M_{uy}} = \frac{5}{2} \quad \text{and} \quad \frac{P_x}{P_y} = \frac{6}{2} \quad \text{so} \quad \frac{5}{2} < \frac{6}{2}$$

The  $MRS$  is the slope of the  $I_{dc}$  and the price ratio is the slope of the  $BC$ . The slope of the  $I_{dc}$  is flatter than that of the  $BC$ , since  $MRS < \frac{P_x}{P_y}$ .



Notice that when the  $MRS < \frac{P_x}{P_y}$ , the best  $I_{dc}$  we can reach is only affordable when we consume all  $Y$ . Therefore,  $x^* = 0$  and

$$y^* = \frac{M}{P_y} = \frac{24}{2} = 12$$

Best  $I_{dc}$  Possible.

Method 2: Compare  $\frac{M_{ux}}{P_x}$  vs  $\frac{M_{uy}}{P_y}$

$$\frac{M_{ux}}{P_x} = \frac{5}{6} \quad \text{and} \quad \frac{M_{uy}}{P_y} = \frac{2}{2} \quad \text{so} \quad \frac{5}{6} < 1$$

A dollar spent on  $x$  gives  $\frac{5}{6}$  of a util, while a dollar spent on  $y$  gives a util.  $y$  is better, so spend all  $M$  on  $y$ .

$$x^* = 0$$

$$y^* = \frac{M}{P_y} = \frac{24}{2} = 12$$

Method 3: Compare utility of all x vs All y

All x

$$x = \frac{m}{P_x} = 4$$

$$u(x, y) = 5(4) + 2(0) = \underline{\underline{20}}$$

All y

$$y = \frac{m}{P_y} = 12$$

$$u(x, y) = 5(0) + 2(12) = \underline{\underline{24}}$$

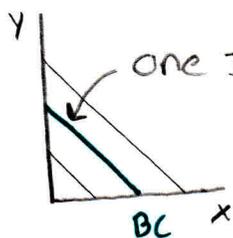
y gives more utility, so consume all y.  $x^* = 0 \quad y^* = 12$

2)  $u(x, y) = 2x + 3y \quad P_x = 2 \quad P_y = 3 \quad m = 12 \quad \text{Find } x^* \text{ and } y^*$

method 1:

$mrs = \frac{2}{3}$  and  $\frac{P_x}{P_y} = \frac{2}{3}$  so  $mrs = \frac{P_x}{P_y}$  what does this mean?

Graphically:



one  $I_{dc}$  intersects the BC perfectly. This is the highest  $I_{dc}$  we can afford. Every point on this  $I_{dc}$  is Indifferent (equal in utility) and affordable. Therefore, any combination of x and y that is on the Budget line (uses all m) is indifferent (equally good), and is an optimal combination.

Conceptually  $\longrightarrow$

$0 \leq x^* \leq 4$  and  $0 \leq y^* \leq 12$  such that all m (income) is used.

3)  $U(x, y) = 3x + y$  No  $P_x/P_y/m$  given Find  $x^*$  and  $y^*$

Method 1:

$MRS = \frac{3}{1}$  and  $\frac{P_x}{P_y} = \text{unknown}$ , so  $MRS$  could be  $> < = \frac{P_x}{P_y}$

Since we don't know  $\frac{P_x}{P_y}$ , Create a Conditional Statement that tells us how to consume based on the Price ratio.

if  $MRS < \frac{P_x}{P_y}$ , then  $x^* = 0$  and  $y^* = \frac{M}{P_y}$

if  $MRS > \frac{P_x}{P_y}$ , then  $x^* = \frac{M}{P_x}$  and  $y^* = 0$

if  $MRS = \frac{P_x}{P_y}$ , then  $0 \leq x^* \leq \frac{M}{P_x}$  and  $0 \leq y^* \leq \frac{M}{P_y}$ , such that all  $M$  is used.

# Perfect Complements

1)  $U(x, y) = \min(2x, y)$   $P_x = 10$   $P_y = 2$   $m = 28$  Find  $x^*$  and  $y^*$

Comp: Set the interior equal, Plug into BC

$2x = y$  or  $x = \frac{y}{2}$ , so for every unit of  $x$ , we want 2 units of  $y$ .

$$P_x x + P_y y = m \rightarrow P_x \frac{y}{2} + P_y y = m \rightarrow y \left( \frac{1}{2} P_x + P_y \right) = m$$

$$y^* = \frac{m}{\left( \frac{1}{2} P_x + P_y \right)}$$

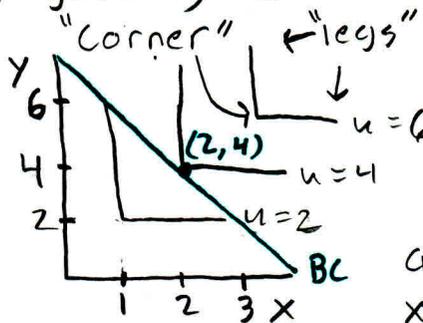
$$x = \frac{y}{2} \rightarrow x^* = \frac{m}{\left( \frac{1}{2} P_x + P_y \right)} \rightarrow x^* = \frac{m}{P_x + 2P_y}$$

$$x^* = \frac{28}{10 + 2(2)} = 2$$

$$y^* = \frac{28}{5 + 2} = 4$$

always wait to plug in variables if possible. This gives you a "general" demand equation. Given any combination of  $P_x/P_y/m$ , you can just plug in to find demand.

↳ notice if we plug this into our utility function we get  $U(x, y) = \min(2(2), 4) = 4$ . The interior is equal so we are not wasting our money getting extra of either good.

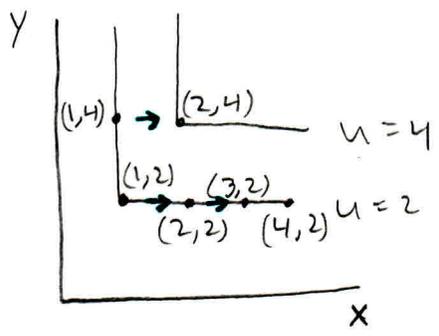


To consume on the "legs" of the Idc we are already on ( $u=4$ ), we would need more  $M$ . The "legs" represent an inefficient combination of  $x$  and  $y$ , since it is more expensive than the corner, but gives the same level of utility. unless  $x$  or  $y$  is fixed, forcing us to take an inefficient combination of  $x$  and  $y$ , we always consume at the corner.

1)  $U(x,y) = \min(2x,y)$   $\downarrow^A$   $\downarrow^B$   $\downarrow^C$   
 a. what is  $m_{ux}$  at (1,2)? At (3,2)? At (1,4)?

Normally to find  $m_{ux}$  or  $m_{uy}$ , we take the derivative of the function in terms of  $x$  or  $y$ . However it is not possible to take the derivative of a min function. Instead, we will do it the old fashion way by subtracting the utility of the end bundle from that of the starting bundle.  $m_{ux}$  = utility increase from one more unit of  $x$ , so the end bundle will be the start bundle + 1  $x$ .

	Utility of end bundle	-	utility of start bundle	=	$m_{ux}$
start	(1,2)		(2,2)		
end					
	$\min(2(2), 2) = 2$	-	$\min(2(1), 2) = 2$	=	0
	$\min(2(4), 2) = 2$	-	$\min(2(3), 2) = 2$	=	0
	$\min(2(2), 4) = 4$	-	$\min(2(1), 4) = 2$	=	2



$m_{ux}$  and  $m_{uy}$  for Perfect Complements depends on how much  $x$  and  $y$  you have.

Notice that for 1 more unit of  $x$  to increase utility, we need extra  $y$  to pair with that  $x$ . Ex: you need 2 units of Jelly ( $y$ ) and 1 unit of Peanut butter to make one Sandwich for A, you have exactly enough ingredients to make a sandwich then you more Peanut butter. Obviously, getting more PB does not help you because you don't have extra Jelly. For B, you have enough to make one sandwich and extra PB, then you get even more PB. Obviously this does not help you. For C, you have enough to make a sandwich and have some Jelly left over when you get extra PB, you can pair it with the extra Jelly to make more sandwiches.

$$2) U(x, y) = 5 \min(x, \frac{y}{5}) \quad P_x = 5 \quad P_y = 1 \quad m = 30 \quad y = 20 \quad \text{Find } x^*$$

Income already spent on  $y \rightarrow P_y \cdot y \rightarrow 1 \cdot 20 = 20$ , so \$10 of income left to spend on  $x$ .

Currently the function is  $5 \min(x, \frac{20}{5})$

We want the interior to be equal, so we want  $x = \frac{20}{5}$

However, we can not afford 4 units of  $x$  with the income we have left, so get as much  $x$  as possible.  $x = 4$

$$x^* = \frac{\text{leftover } m}{P_x} = \frac{10}{5} = 2 \quad \boxed{y = 20 \quad x^* = 2}$$

a) Find  $x^*$  and  $y^*$  if  $y$  is not fixed at 20

Set the interior equal, so  $x = \frac{y}{5}$ . Plug into BC.

$$P_x \frac{y}{5} + P_y y = m \rightarrow y \left( \frac{1}{5} P_x + P_y \right) = m \rightarrow y^* = \frac{m}{\left( \frac{1}{5} P_x + P_y \right)}$$

$$x = \frac{y}{5} \rightarrow x = \frac{m}{\left( \frac{1}{5} P_x + P_y \right)} \rightarrow x^* = \frac{m}{P_x + 5 P_y}$$

$$x^* = \frac{30}{(5) + 5} = 3$$

$$y^* = \frac{30}{\frac{1}{5}(5) + 1} = 15$$

$$\boxed{x^* = 3 \quad y^* = 15}$$

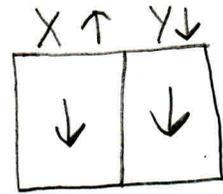
# Cobb-douglas

1)  $u(x, y) = x^2 y$  no prices/Income given Find  $x^*$  and  $y^*$

Cobb-doug: Use Tangency  $\rightarrow$   $mrs = \frac{P_x}{P_y}$ . Prove that you can use tangency by showing that the function has  $dmrs$ .

Proof of  $dmrs$ .  $mrs = \frac{m_{ux}}{m_{uy}} \rightarrow$

$$\frac{2xy}{x^2} \rightarrow \frac{2y}{x} \rightarrow \text{as } x \uparrow, mrs \downarrow. \text{ As } y \downarrow, mrs \downarrow. \text{ Can use Tangency.}$$



$$\frac{2y}{x} = \frac{P_x}{P_y} \rightarrow y = \frac{P_x x}{2P_y} \quad \left( \text{This is not } y^* \text{ because } x, \text{ another variable, is still in the equation} \right)$$

Plug into BC

$$P_x x + P_y \left( \frac{P_x x}{2P_y} \right) = m \rightarrow x \left( P_x + \frac{P_x}{2} \right) = m \rightarrow$$

$x^* = \frac{m}{\left( P_x + \frac{P_x}{2} \right)}$

Plug  $x^*$  into the tangency condition to find  $y^*$ .

$$y = \frac{P_x x}{2P_y} \rightarrow y = \frac{P_x \left( \frac{m}{P_x + \frac{P_x}{2}} \right)}{2P_y} \rightarrow y = \frac{P_x \left( \frac{m}{\frac{3P_x}{2}} \right)}{2P_y}$$

$$y = \frac{m}{\frac{6P_y}{2}} \rightarrow$$

$y^* = \frac{m}{3P_y}$

Side note:  $m_{ux} = 2xy$   $m_{uy} = x^2$ . notice the marginal utility of  $x$  is dependent on how much  $y$  you have, and the  $m_{uy}$  depends on the level of  $x$ . If we have a lot of  $x$  and a little  $y$ , one more unit of  $y$  is very valuable!

1)

a) What proportion of income do we spend on  $x$  and  $y$  if  $u(x, y) = x^2 y$  and  $u(x, y) = x^{\frac{1}{2}} y$ ?

Use the Cobb-Douglas exponent trick: if the exponents in a Cobb-Douglas function add up to 1, the fraction tells you how much income you spend on each good. For Ex,  $u(x, y) = x^{\frac{1}{2}} y^{\frac{1}{2}}$ . Spend half of your income on  $x$ , and half on  $y$ . If the exponents do not add up to 1, use monotonic transformations until they do.

$u(x, y) = x^2 y \rightarrow (x^2 y)^{\frac{1}{3}} \rightarrow x^{\frac{2}{3}} y^{\frac{1}{3}} \rightarrow$  spend  $\frac{2}{3}$  of income on  $x$ , and  $\frac{1}{3}$  on  $y$ . If  $m = 30$ ,  $P_x = 1$ ,  $P_y = 1$ , we will spend \$20 on  $x$  and \$10 on  $y$ , so  $x^* = 20$  and  $y^* = 10$ . Try it if you don't believe me!

$u(x, y) = x^{\frac{1}{2}} y \rightarrow (x^{\frac{1}{2}} y)^2 \rightarrow x y^2 \rightarrow (x y^2)^{\frac{1}{3}} \rightarrow x^{\frac{1}{3}} y^{\frac{2}{3}} \rightarrow$  spend  $\frac{1}{3}$  of income on  $x$  and  $\frac{2}{3}$  of income on  $y$ .

## Quasi-linear convex (decreasing)

1)  $U(x, y) = 2x^{\frac{1}{2}} + y$  No Prices/Income what is  $x^*$  and  $y^*$ ?

Quasi-linear: Test to prove dmrs, then use tangency.

Quasi-linear decreasing always has dmrs, but quasi-linear increasing has Imrs (Increasing).

$$\text{mrs} = \frac{x^{-\frac{1}{2}}}{1} = x^{-\frac{1}{2}} \quad \text{(same as } \frac{1}{x^{\frac{1}{2}}})$$

$x \uparrow$	$y \downarrow$
$\downarrow$	-

So dmrs  $\checkmark$   
use Tangency.

Tangency:  $x^{-\frac{1}{2}} = \frac{p_x}{p_y} \rightarrow \boxed{x^* = \frac{p_y^2}{p_x^2}}$

To find  $y^*$ , Plug  $x^*$  into the BC  $\rightarrow$

$$p_x \left( \frac{p_y^2}{p_x^2} \right) + p_y y = m \rightarrow p_y y = m - \frac{p_y^2}{p_x} \rightarrow \boxed{y^* = \frac{m}{p_y} - \frac{p_y}{p_x}}$$

$$1) U(x, y) = 2x^{\frac{1}{2}} + y \quad x^* = \frac{Py^2}{Px^2} \quad y^* = \frac{m}{Py} - \frac{Px}{Py}$$

a) what is  $x^*, y^*$  if  $m = 30$   $Py = 5$   $Px = 1$ ?

$$x^* = \frac{5^2}{1^2} = 25 \quad y^* = \frac{30}{5} - \frac{5}{1} = 6 - 5 = 1$$

$$\boxed{x^* = 25 \quad y^* = 1}$$

b) what is  $x^*, y^*$  if  $m = 3$   $Py = 5$   $Px = 1$ ?

$$x^* = \frac{25}{1} = 25 \quad y^* = \frac{3}{5} - \frac{5}{1} = -4\frac{2}{5}$$

Although our demand functions for  $x^*$  and  $y^*$  are correct, the answer we got is obviously not. We want 25 units of  $x$ , but can't afford it, so we compensate by getting negative amounts of  $y$ , which is impossible.

What is happening here?  $x^* = \frac{Py^2}{Px^2}$ , notice there is no  $m$  in the demand equation.  $x$  is a zero income effect good, meaning the amount of  $x$  we want does not depend on our income. Sometimes we will not have enough income to afford the amount of the zero-income effect good we want. We must get all of the zero-income effect good that we want before we get any of the other good. Since we can only afford 3 units of  $x$ ,  $\boxed{x^* = 3 \quad y^* = 0}$

in this example,  $x$  would be cooking supplies (pots, knives, silverware) and  $y$  would be ingredients. We want a certain amount of cooking supplies before we get any ingredients. (You can't cook mac and cheese without a pot).

## Quasi-linear Concave (Increasing)

1)  $U(x, y) = x^2 + 5y$       $P_x = 5$     $P_y = 3$     $m = 30$    Find  $x^*$  and  $y^*$

Quasi-linear: Test to prove Increasing MRS (if dMRS, you have a quasi-linear convex, not concave). Consume all of  $x$  or all of  $y$ , whichever gives more utility.

All  $x$

$$x = \frac{m}{P_x} = 6$$

$$U(x, y) = 6^2 + 5(0) = \underline{\underline{36}}$$

All  $y$

$$y = \frac{m}{P_y} = 10$$

$$U(x, y) = (0)^2 + 5(10) = \underline{\underline{50}}$$

$$\begin{aligned} x^* &= 0 \\ y^* &= 10 \end{aligned}$$

spend All  $m$  on  $y$

## max

1)  $U(x, y) = 2 \max(3x, y)$       $P_x = 4$     $P_y = 1$     $m = 20$    Find  $x^*$  and  $y^*$

max: Choose all of the good that gives the most utility.

All  $x$

$$x = \frac{m}{P_x} = 5$$

$$U(x, y) = 2 \max(3(5), 0) = \underline{\underline{30}}$$

All  $y$

$$y = \frac{m}{P_y} = 20$$

$$U(x, y) = 2 \max(3(0), 20) = \underline{\underline{40}}$$

$$\begin{aligned} x^* &= 0 \\ y^* &= 20 \end{aligned}$$

Interesting side note:

